String Theory, Supersymmetry and Geometry

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Introduction

Close and fruitful interplay between

String Theory $\Leftrightarrow$ Supersymmetry $\Leftrightarrow$ Geometry

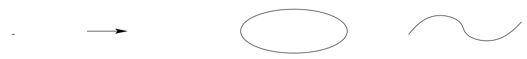
purpose of this talk:

- review some of its aspects
- discuss string compactifications on manifolds with $SU(3)$-structure

work in collaboration with M. Grana, S. Gurrieri, A. Micu, D. Waldram

String Theory

basic idea: point-like objects $\longrightarrow$ extended objects (strings)



Strings move in 10-dimensional space-time background

contact with "our world": Compactification

$\Rightarrow$ space-time background:

$$\mathcal{M}_{10} = R_{1,3} \times Y_6$$

$R_{1,3}$:	four-dimensional ($d = 4$) Minkowski-space
Y_6:	compact manifold – determines amount of supersymmetry

Different string theories:

Type I, Type II, Heterotic

they differ in spectrum of excitations and their interactions

talk today:

focus only on Type II string theories

they come in two versions: IIA & IIB – both are supersymmetric

massless spectrum in $d = 10$:

	IIA	IIB
NS:	$G_{MN},\ H_3 = dB_2,\ \Phi$	$G_{MN},\ H_3 = dB_2,\ \Phi$
RR:	$F_2 = dC_1,\ F_4 = dC_3$	$l,\ F_3 = dC_2,\ F_5^* = dC_4$

F_p $= p$-form field strength

C_{p-1} $= (p-1)$-form gauge potential

Compactification: determine Y_6

Lorentz group on space-time background $\mathcal{M}_{10} = R_{1,3} \times Y_6$ decomposes

$$SO(1,9) \to SO(1,3) \times SO(6)$$

spinor decompose accordingly:

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}})$$

impose two conditions:

1. demand that supercharge Q exist $\Rightarrow$ structure group of Y_6 has to be reduced

$$SO(6) \to SU(3) \quad \text{s.t.} \quad \mathbf{4} \to \mathbf{3} + \mathbf{1}$$

$\Rightarrow$ invariant spinor η exists $\Rightarrow$ Y_6 has $SU(3)$-structure

2. background preserves supersymmetry

$$\delta\Psi_M = \nabla_M \eta + (\gamma \cdot F)_M = 0$$

$\Rightarrow$ for $F = 0$: $\nabla\eta \;=\; 0$ $\Rightarrow$ Y_6 is Calabi-Yau manifold

Calabi-Yau Threefold Y

- Levi-Civita connection has $SU(3)$ holonomy $\Rightarrow$ Kähler manifold
- integrability condition: $R_{ij} = 0 \qquad \Rightarrow$ Ricci-flat manifold
- existence of invariant spinor η implies existence of two invariant tensors:
  - complex structure

    $$J = \eta^\dagger \gamma\gamma\eta \;, \qquad J^2 = -1$$

    corresponding $(1,1)$-form J is closed

    $$dJ = 0$$

  - $(3,0)$-form

    $$\Omega = \eta^\dagger \gamma\gamma\gamma\eta \;, \qquad d\Omega = 0$$

Mirror Symmetry

conjecture:

for ‘every’ Y there exists a mirror manifold $\tilde{Y}$ with

$$h^{1,1}(Y) = h^{1,2}(\tilde{Y}) \ , \qquad h^{1,2}(Y) = h^{1,1}(\tilde{Y})$$

manifestation in string theory:

IIA in background $R_{1,3} \times Y$ $\equiv$ IIB in background $R_{1,3} \times \tilde{Y}$

Kaluza-Klein compactification in space-time background: $R_{1,3} \times Y_6$

- massless scalars

$$\Delta_{10}\phi = (\Delta_4 + \Delta_6)\phi = (\Delta_4 + m^2)\phi = 0$$

$\Rightarrow$ massless $d = 4$ spectrum $=$ zero modes of Δ_6 $=$ harmonic forms in $H^{(p,q)}(Y)$

- Hodge numbers: $h^{p,q} = \dim H^{p,q}(Y)$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{1,2} & & h^{1,2} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

- deformations of Calabi-Yau metric form geometrical moduli space

$$\mathcal{M} = \mathcal{M}_{\rm cs}^{h^{(1,2)}} \times \mathcal{M}_{\rm k}^{h^{(1,1)}}$$

cs: deformations of complex structure, k: deformations of Kähler form

appear as scalar fields in effective action: supergravity in $d = 4$

Scalar fields in supergravity

$$\mathcal{L} = \frac{1}{2} R - g_{ab}(z)\, \partial_\mu z^a \partial^\mu z^b + \ldots$$

Supersymmetry constrains σ-model metric g_{ab}:

supersymmetry	geometry of scalar manifold
$N = 1$	Kähler
$N = 2$ global	special Kähler $\times$ Hyperkähler
$N = 2$ local	special Kähler $\times$ quaternionic Kähler
$N = 4$	$\frac{SO(6+n,n)}{SO(6+n)\times SO(n)}$
$N = 8$	$\frac{E_{7,7}}{SU(8)}$

Compactifications of type II string theory on Calabi-Yau manifolds

low energy effective action: $N = 2$ supergravity

- scalar manifold:

$$\begin{aligned} \text{IIA}: &\qquad \mathcal{M} = \mathcal{M}^{SK}_{2h^{(1,1)}} \times \mathcal{M}^{QK}_{4(h^{(1,2)}+1)} \\ \text{IIB}: &\qquad \mathcal{M} = \mathcal{M}^{SK}_{2h^{(1,2)}} \times \mathcal{M}^{QK}_{4(h^{(1,1)}+1)} \end{aligned}$$

- $\mathcal{M}^{SK}_{2h^{(1,2)}}$:

$$g_{ab} = \frac{\partial}{\partial z^a}\frac{\partial}{\partial \bar{z}^b}\, K_{\text{cs}}\ , \qquad K_{\text{cs}} = -\ln i\Big[\bar{X}^A(\bar{z})F_A(z) - X^B(z)\bar{F}_B(\bar{z})\Big]\ .$$

$$\text{Periods of } (3,0)-\text{form } \Omega: \qquad X^A(z) = \int_{\gamma_A}\Omega\ , \qquad F_B(z) = \int_{\gamma_B^*}\Omega$$

- $\mathcal{M}^{Q}_{4(h^{(1,1)}+1)}$ has $\mathcal{M}^{SK}_{2h^{(1,1)}}$ as a subspace

  $\Rightarrow$ also determined by holomorphic prepotential F

- both F's known exactly by mirror symmetry

Generalization: manifolds with $SU(3)$ structure

Recall that we imposed two conditions:

1. demand that two supercharges Q exist

   $\Rightarrow$ structure group of Y_6 has to be reduced $SO(6) \to SU(3)$

   $\Rightarrow$ invariant spinor η exists $\Rightarrow$ Y_6 has $SU(3)$-structure

2. background preserves supersymmetry $\delta\Psi_M = \nabla_M \eta + (\gamma \cdot F)_M = 0$

   $\Rightarrow$ for $F = 0$: $\quad \nabla\eta \; = \; 0 \quad \Rightarrow$ Y_6 is Calabi-Yau manifold

Generalizations: insist on 1. (existence of Q) but relax

$$(i) \qquad F \neq 0 \,, \qquad \nabla\eta \neq 0 \qquad \text{but} \qquad \delta\Psi_M = 0$$

corresponds to supersymmetric background with non-trivial flux

$$(ii) \qquad F \neq 0 \,, \qquad \nabla\eta \neq 0 \qquad \text{and} \qquad \delta\Psi_M \neq 0$$

corresponds to spontaneously broken supersymmetry possible situations:

- $F \neq 0$: Y_6 has non-trivial background flux
- $\nabla\eta \neq 0$: Y_6 is manifold of $SU(3)$ structure with torsion

  [Gray, Hervella, Salamon, Falcitelli, Farinola, Chiossi, Friedrich, Ivanov, Hitchin, ...]

  such manifolds are characterized by existence of invariant spinor η which obeys

$$\nabla^{(T)}\eta \;\equiv\; (\nabla^{(LC)} + T_0)\,\eta \;=\; 0 \;, \qquad T_0 : \text{intrinsic (con)-torsion}$$

  $\Rightarrow$ existence of two invariant tensors:

$$\text{almost complex structure} \quad J = \eta^\dagger\gamma\gamma\eta \;, \quad J^2 = -1,$$
$$(3,0)\text{-form} \quad \Omega = \eta^\dagger\gamma\gamma\gamma\eta$$

  generically:

$$dJ \neq 0 \;, \qquad d\Omega \neq 0$$

  obstructed by T_0

  $\Rightarrow$ manifolds are not complex, not Kähler, not Ricci-flat

Compactifications on manifolds with $SU(3)$ structure [Grana,Gurrieri,Micu,Waldram,JL]

➪ insist on only two gravitinos in gravitational multiplet

$\Rightarrow$

$$d\Omega \neq 0 \,, \qquad dJ \neq 0 \,, \qquad d\,(J \wedge J) = 0$$

➪ choice of $d\Omega$, dJ (choice of the torsion) leads to family of manifolds

$$h^{(2)}(\hat{Y}) = h^{(1,1)}(Y) - n \,, \qquad h^{(3)}(\hat{Y}) = h^{(3)}(Y) - 2n$$

➪ $N = 2$ implies scalar manifold: $\mathcal{M} = \mathcal{M}^{SK} \times \mathcal{M}^{Q}$

➪ $\mathcal{M}^{SK}$:

$$K = -\ln i \Big[\bar{X}^A(\bar{z}) F_A(z) - X^B(z) \bar{F}_B(\bar{z}) \Big] \,.$$

$$X^A(z) = \int_{\gamma_A} \Omega \,, \qquad F_B(z) = \int_{\gamma^*_B} \Omega$$

$\Rightarrow$ $\mathcal{M}^{SK}$ determined by holomorphic F as required by $N = 2$ supergravity

➪ generalized geometrical mirror symmetry $\Rightarrow$ $\mathcal{M}^{Q}$ also determined by F.

Conclusions / Open Questions

- Interplay string theory $\Leftrightarrow$ Supersymmetry $\Leftrightarrow$ Geometry
- Supersymmetry determines properties of compactification
  $\Rightarrow$ compactification on manifolds with $SU(3)$ structure
- Geometrical moduli appear as scalar fields in effective action
  scalar manifold constrained by supersymmetry
- Manifolds with $SU(3)$ structure obey this constraint
  and geometry of scalar manifold can be explicitly computed
- explicit construction of manifolds with torsion ?
- mathematical definition of moduli space ?